

MAC-CPTM Situations Project

Situation 09: Perfect Square Trinomials

Prepared at University of Georgia
Center for Proficiency in Teaching Mathematics
May 2005 – Bob Allen and Dennis Hembree

Edited at University of Georgia
Center for Proficiency in Teaching Mathematics
27 June 2006 – Sarah Donaldson
9 December 2008 – Brian Gleason and Shawn Broderick

Prompt

While teaching about factoring perfect square trinomials, a teacher realizes that the students have the impression that a trinomial is a perfect square trinomial if and only if the first and last terms of the trinomial are perfect squares. That is, they believe that the ‘middle term’ is irrelevant. The teacher would like to construct a counterexample on the spot.

Commentary

The teacher must be able to recognize and factor perfect square trinomials, which entails understanding the properties of the components of the trinomial that identify it as a perfect square trinomial. Then, the teacher must be able to produce a trinomial whose components lack those properties.

Mathematical Foci

Mathematical Focus 1

A trinomial is so named because it consists of three parts or monomials. Factorable trinomials (over a given ring) exhibit a certain structure where one of the three monomials is composed of factors of the other two monomials. It is generally placed in between the other two, and so is referred to as the ‘middle term.’ In a perfect square trinomial, the monomials have additional properties.

The product of the generalized factors $(a \pm b)(x \pm y)$ is $ax \pm ay \pm bx \pm by$. If this is to be a trinomial, precisely two of these terms must be *like terms*, meaning that their sum can be simplified. An example of this would be for $ay = nbx$ for some $n \in \mathbb{Z}$, for then the polynomial is simplified to be $ax \pm (n + 1)bx \pm by$, which is a factorable trinomial.

A perfect square trinomial is further distinguished by the fact that its non-middle terms are perfect square monomials, and that its middle term equals twice the product of the

principal square roots of the other two terms. To see this, form a perfect square trinomial by squaring the factor $(a + b)$ as follows:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= a \cdot a + a \cdot b + b \cdot a + b \cdot b \\ &= a^2 + 2ab + b^2\end{aligned}$$

Mathematical Focus 2

The property of perfect square trinomials explained in Focus 1 can be used to help one identify and quickly factor them.

The middle term of a perfect square trinomial is $2ab$ (see Focus 1). This indicates that the middle term is twice the product of a and b , the principal square roots of the first and last terms of the trinomial. This implies that trinomial whose middle term is not equal to twice the product of the principal square roots of the first and last terms is not a perfect square trinomial.

For example, consider $16x^2 + 24xy + 9y^2$.

In this case $\sqrt{16x^2} = 4x$, $\sqrt{9y^2} = 3y$, and $24xy = 2(4x)(3y)$.

Therefore if we say that $4x = a$ and that $3y = b$, then $16x^2 + 24xy + 9y^2 = a^2 + 2ab + b^2$, thereby confirming that it is a perfect square trinomial and can be factored as $(4x + 3y)^2 = (a + b)^2$.

It is possible to construct a trinomial whose non-middle terms are perfect square monomials, but whose middle term lacks the property explained above. A trinomial such as $16x^2 + 23xy + 9y^2$ is not a perfect square trinomial because although the first and last terms are perfect squares, the middle term, $23xy$, is not twice the product of the principal square roots of the first and last terms of the trinomial.

Thus, in order to produce a trinomial that will demonstrate that the middle term matters when factoring perfect square trinomials, the teacher must simply write a trinomial in the form $a^2 + p + b^2$, where $p \neq 2ab$.

Post-Commentary

There are many subtleties of factoring polynomials that are usually ignored, including in the above discussion. For example, factoring occurs over a ring, and this ring will determine which polynomials are factorable and which are prime. In secondary curricula, polynomials are usually factored over \mathbb{Z} , but this is by convention. Additionally, the ‘variables’ used in polynomials are often not explicitly stated. For example, it is implicitly assumed that in the trinomial $ax^2 + bx + c$ that x is the indeterminate in some ring $R[x]$ and that a , b , and c are parameters taken from the ring R . Contrast this with the polynomial discussed throughout this situation, $a^2 + 2ab + b^2$, where a and b are treated more as generalized numbers than parameters. These and other subtleties of

factorization and of polynomials make the problem of learning to identify and factor a certain category of polynomials rather non-trivial.